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A new three -Parameters Inverse Power Exponentiated Pareto Distribution: Properties and its Applications

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Abstract: The topic of the article is proposing the inverse power Exponentiated Pareto, a new three-parameter lifetime distribution. The proposed distribution is obtained as the inverse form of the power Exponentiated Pareto distribution. Some statistical properties of the inverse power Exponentiated Pareto model are introduced. The model parameters are estimated based on maximum likelihood. To evaluate the performance of estimators based on their mean square errors, an extensive simulation is conducted. The proposed model's superiority over some well-known distributions are observed using two sets of real-life data. The observations show that the proposed model can produce a better fit than some well-known distributions.

keywords: Exponentiated Pareto; Moments; Entropy; Quantile function; Order Statistic; Stress-Strength Parameter; Maximum likelihood estimator**.**

1.Introduction

The Pareto distribution is a very simple and flexible model, having features of accommodating various types. Besides providing a suitable model for typical income and wealth data through some more flexible and generalized variants of the classical Pareto distribution, these are found very useful in various problems related to life testing, survival analysis, telecommunication, actuarial science, economics and finance etc. In statistical literature, the Pareto distribution has been widely used as a model to analyze a variety of socioeconomic issues. In fact, the Pareto distribution and its generalization give a very flexible family of heavy-tailed distributions that may be used to model income distributions as well as a wide range of other distributions associated with social and economic problems. For a more extensive discussion on the use of these models in the context of income distributions, see Villaseñor and Arnold [1]. Pal et al. [2] and Ali et al. [3] analyzed and discussed the properties of a variety of exponentiated distributions, including the exponentiated Pareto distribution. They illustrated that the distribution of NASDAQ data from the American stock exchange fits perfectly with exponentiated Pareto distribution. Some more exclusive examples where the Pareto distributions provides good fit are the standardized price returns on individual stocks, the width of human settlements, the volume of oil reserves in oil field etc. It was made by adding an exponentiating parameter to the cumulative distribution function(cdf) of a Pareto distribution. The Exponentiated Pareto distribution (EPD) was proposed by Gupta et al. [4] as an effective method for evaluating a variety of lifetime data.

Exponentiating the cumulative distribution function (cdf) of an existing model is the basic idea of exponentiated distribution. Due to the additional parameter, it gives the conventional models more flexibility. For estimating the location and scale parameters of the EPD, mahmoud et al. [5] depend on progressive type-II right censored order statistics and maximum likelihood estimators (MLEs). The EP distribution has a very flexible structure as a result of its decreasing or upside-down bathtub form failure rates depending on shape parameters. This characteristic gives advantages for modelling extreme phenomena, especially for hydrology, see Chen and Cheng [6]. Additionally, the EP distribution's heavier or lighter-tailed properties make it a practical alternative to the exponential distribution. Afify [7] presented the conventional and Bayesian inferences for EP distribution under different censoring approaches. Al-Omari et al. [8] also analyzed the stress-strength reliability for EP distribution based on several sampling methods. They illustrated how the Exponentiated Pareto distribution and the Nasdaq data tail distribution fit each other well.

The probability density function (pdf) of a random variable X has the Exponentiated Pareto distribution given by:

$$f_{EP}(x,\alpha,\beta) = \alpha\beta(1+x)^{-(\beta+1)} \left(1-(1+x)^{-\beta}\right)^{\alpha-1};\ x,\alpha,\beta > 0, \qquad (1)$$

where the scale parameter is λ and the shape parameters are α and β. According to Eq. (1), the cumulative distribution function (cdf) gives as follows:

$$F_{EP}(x,\alpha,\beta) = \left(1-(1+x)^{-\beta}\right)^{\alpha};\ x,\alpha,\beta > 0. \qquad (2)$$

when α=1, the Exponentiated Pareto distribution (EPD) reduces to the standard Pareto distribution of the second kind with C=1 and a=λ, see Johnson et al. [9].

In this paper, we modify the power exponentiated Pareto distribution using an inverse scheme. In order to construct the inverse power Exponentiated Pareto (IP Exponentiated Pareto) distribution, we specifically rely on the random variable Z=(1/γ) distribution. The literature has investigated several inverses of well-known distributions, showing characteristics that are distinct from those of the base distributions.

On the other hand there are many researcher who are interested to expand a modification of the power of distributions are, Hassan and Abd-Allah [10] introduced the inverse power Lomax distribution, Ghitany et al. [11] presented power Lindley distribution and associated inference, the statistical theory and application of the inverse power Muth distribution was presented by Chesneau and Agiwal [12] and Afify et al. [13] proposed a new two-parameter Burr-Hatke distribution, as well as the characteristics and applications both in Bayesian and non-Bayesian inference. The inverse Exponentiated Pareto distribution (IPEP) distribution is a new three-parameter distribution that we first introduce. The IPEP distribution's statistical characteristics are provided. Second, using two real data sets, we consider parameter estimates for IPEP distribution, with the first data set representing growth hormone. Growth hormone deficiency was identified in children of the Program Hormonal (de Crescimento da Secretaria da Sa'ude de Minas Gerais), and the second set of data represents the Susquehenna River's maximum flood levels (measured in million cubic feet per second) at Harrisburg, Pennsylvania, between 1890 and 1969.

This Paper can be designed as follows. In Sect. 2 describes how the IPEP distribution is constructed. In Sect. 3 some statistical properties are discussed. In Sect.4 estimation of the parameters is derived. In Sect. 5 Simulation study is carried out to provide an illustration of theoretical results. In Sect. 6, the significance of the IPEP model is shown using real data sets followed by concluding remarks.

2 Inverse Power Exponentiated Pareto Model

In this section, we introduce and study the inverse power Exponentiated Pareto (IPEP) distribution. Mathematically, the distribution is defined with the following pdf:

$$f_{IPEP}(x,\alpha,\beta,\lambda) = \alpha\beta\lambda x^{-\lambda-1}\left(1+x^{-\lambda}\right)^{-\beta-1}\left(1-\left(1+x^{-\lambda}\right)^{-\beta}\right)^{\alpha-1};\ x,\alpha,\beta,\lambda > 0, \qquad (3)$$

where α and β are the shape parameters and λ is the scale parameter.

The cdf corresponding to Eq. (3) is defined by

$$F_{IPEP}(x,\alpha,\beta,\lambda) = 1-\left(1-\left(1+x^{-\lambda}\right)^{-\beta}\right)^{\alpha};\ x,\alpha,\beta,\lambda > 0, \qquad (4)$$

Figs. 1 and 2 show several graphs of the distribution's pdf and cdf for various parameter values. The pdf of the IPEP distribution is quite flexible and can assume several forms, in Figs. 1 and 2.

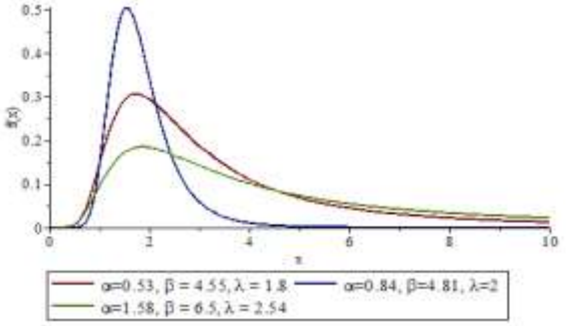

Fig. 1: The pdf of the IPEP distribution for some parameter values.

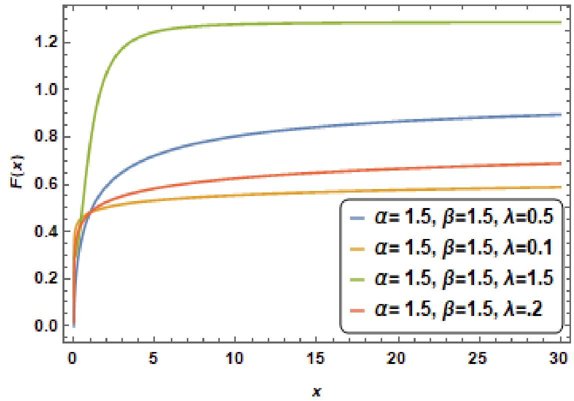

Fig. 2: The cdf of the IPEP distribution for some parameter values.

In the IPEP distribution, the hazard rate function (hrf) and the survival function are provided, respectively, by

$$h_{IPEP}(x,\alpha,\beta,\lambda)=\frac{f(x)}{S(x)}=\frac{\alpha\beta\lambda x^{-\lambda-1}\left(1+x^{-\lambda}\right)^{-\beta-1}\left(1-\left(1+x^{-\lambda}\right)^{-\beta}\right)^{\alpha-1}}{\left(1-\left(1+x^{-\lambda}\right)^{-\beta}\right)^{\alpha}};\ x,\alpha,\beta,\lambda>0,\qquad (5)$$

and

$$S_{IPEP}(x,\alpha,\beta,\lambda)=\left(1-\left(1+x^{-\lambda}\right)^{-\beta}\right)^{\alpha};\ x,\alpha,\beta,\lambda>0.\qquad (6)$$

Figs. 3 and 4 illustrates plots of the IPEP hazard function and survival function some specified values of the parameters.

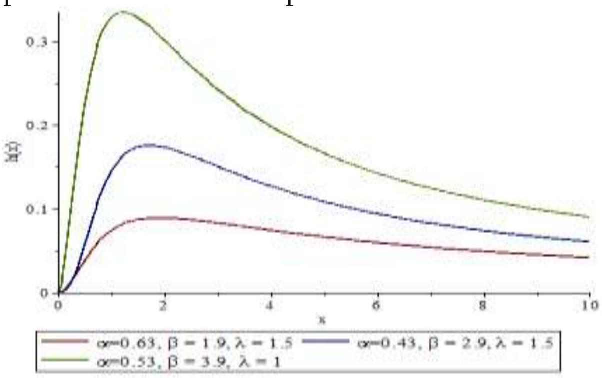

Fig. 3: The hazard function of the IPEP distribution for some parameter values.

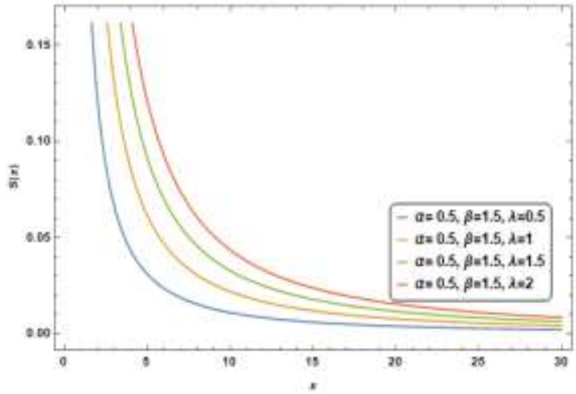

Fig. 4: The survival function of the IPEPdistribution for some parameter values.

3. Some Structural Properties

Here, some statistical characteristics of the IPEP distribution such as, quantile function, ordinary incomplete momments, the probability weighted momment, momment generating function, Re'nyi entropy, order statistics and stress strength parameter are obtained.

3.1 Quantile function

The quantile function in probability and statistics provides the value of a random variable for a probability distribution such that the probability of the variable being less than or equal to that value equals the provided probability. The quantile function of x, where x~IPEP (x; α, β, λ >0,) by putting $F_{IPEP}(x,\alpha,\beta,\lambda)=U$ as

$$1-\left(1-\left(1+x^{-\lambda}\right)^{-\beta}\right)^{\alpha}=U,$$

then

$$x=\left(\left(1-(1-U)^{\frac{1}{\alpha}}\right)^{\frac{-1}{\beta}}-1\right)^{\frac{-1}{\lambda}}.\qquad (7)$$

The random variable X=Q(U) is given by Eq. (7), if U is a uniform variate on the unit interval (0,1).

3.2 Ordinary and incomplete momment

Ordinary moments can be used to obtain many of a distribution's basic properties and characteristics. Assuming that X is a random variable with IPEP and the parameters α,β and λ, it is simple to calculate the rth incomplete moment of x from pdf Eq.(5) as illustrated below

$$E(x^r) = \sum_{i=0}^{\alpha-1} -\alpha\beta(-1)^i \binom{\alpha-1}{i} B\left(\frac{-r}{\lambda} + 1, \beta(i+1) - \frac{r}{\lambda} + 1\right). \quad (8)$$

Proof

$$E(x^r) = \int_0^\infty x^r f_{IPEP}(x,\alpha,\beta,\lambda)dx$$

then

$$E(x^r) = \int_0^\infty x^r \alpha\beta\lambda x^{-\lambda-1}(1+x^{-\lambda})^{-\beta-1} \left(1-(1+x^{-\lambda})^{-\beta}\right)^{\alpha-1} dx$$

by using binomial expansion

$$\left(1-(1+x^{-\lambda})^{-\beta}\right)^{\alpha-1} = \sum_{i=0}^{\alpha-1} \alpha\beta(-1)^i \binom{\alpha-1}{i} (1+x^{-\lambda})^{-\beta i},$$

then

$$E(x^r) = \sum_{i=0}^{\alpha-1} \alpha\beta(-1)^i \binom{\alpha-1}{i}.$$

Thus, the computation of this integration

$$\int_0^\infty x^{r-\lambda-1}(1+x^{-\lambda})^{\beta(i+1)-1}dx = \frac{-1}{\lambda} B\left(\frac{-r}{\lambda} + 1, \beta(i+1) - \frac{r}{\lambda} + 1\right),$$

where$\left(\frac{-r}{\lambda} + 1, \beta(i+1) - \frac{r}{\lambda} + 1\right)$ incomplete Beta function.

This completes the proof.

3.3.Theprobabilityweightedmomments(PWMs)

The PWMs can be defined for any random variable whose ordinary moments exist. They are expectations of particular functions of a random variable. When the inverse form of an extended distribution cannot be explicitly described, the PWMs approach can be used to estimate the parameters and quantiles of the distribution. According to IPEP, the *(r,s)th* PWMs of *X* following the IPEP distribution.

Let's $\nu_{r,s}$, is defined by

$$\nu_{r,s} = \sum_{i=0}^{s} \sum_{j=0}^{\alpha+\alpha_i-1} -\alpha\beta\lambda\,(-1)^{i+j} \binom{s}{i} \binom{\alpha+\alpha_i-1}{j} B\left(\frac{-r}{\lambda} + 1, \beta(j+1) + \frac{r}{\lambda}\right). \quad (9)$$

Proof

$$\nu_{r,s} = \int_{-\infty}^{\infty} x^r f_{IPEP}(x;\alpha,\beta,\lambda)[F_{IPEP}(x;\alpha,\beta,\lambda)]^s dx,$$

then

$$\nu_{r,s} = \int_{-\infty}^{\infty} x^r \alpha\beta\lambda x^{-\lambda-1}(1+x^{-\lambda})^{-\beta-1}\left(1-(1+x^{-\lambda})^{-\beta}\right)^{\alpha-1} \left[1-\left(1-(1+x^{-\lambda})^{-\beta}\right)^{\alpha}\right]^S dx.$$

By using binomial expansion

$$\left[1-\left(1-(1+x^{-\lambda})^{-\beta}\right)^{\alpha}\right]^S = \sum_{i=0}^{s} (-1)^i \binom{s}{i} \left(1-(1+x^{-\lambda})^{-\beta}\right)^{\alpha i},$$

then

$$\nu_{r,s} = \sum_{i=0}^{s} \alpha\beta\lambda(-1)^i \binom{s}{i} \int_0^\infty x^{r-\lambda-1}(1+x^{-\lambda})^{-\beta-1}\left(1-(1+x^{-\lambda})^{-\beta}\right)^{\alpha+\alpha i-1} dx,$$

by using binomial expansion

$$\left(1-(1+x^{-\lambda})^{-\beta}\right)^{\alpha+\alpha i-1} = \sum_{j=0}^{\alpha+\alpha_i-1} (-1)^j \binom{\alpha+\alpha_i-1}{j} (1+x^{-\lambda})^{-\beta j},$$

then

$$\nu_{r,s} = \sum_{i=0}^{s} \sum_{j=0}^{\alpha+\alpha_i-1} \alpha\beta\lambda\,(-1)^{i+j} \binom{s}{i} \binom{\alpha+\alpha_i-1}{j} \int_0^\infty x^{r-\lambda-1}(1+x^{-\lambda})^{-\beta(j+1)-1}dx.$$

Thus, the computation of this integration

$$\int_0^\infty x^{r-\lambda-1}(1+x^{-\lambda})^{-\beta(j+1)-1}dx = \frac{-1}{\lambda} B\left(\frac{-r}{\lambda} + 1, \beta(j+1) + \frac{r}{\lambda}\right).$$

This completes the proof.

3.4 Momment genrating function

Only negative values of its variable t are defined for the moment generating function (mgf) $M_x(t)$ that corresponds to a random variable X with IPEP and parameters α, β and λ. Following that, the momment generating function of X is given by:

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} \int_0^\infty x^j f_{IPEP}(x;\alpha,\beta,\lambda)dx$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} E(x^j)$$

$$= \sum_{j=0}^{\infty} \sum_{i=0}^{\alpha-1} -\alpha\beta(-1)^i \binom{\alpha-1}{i} \frac{t^j}{j!} B\left(\frac{-j}{\lambda} + 1, \beta(i+1) - \frac{j}{\lambda} + 1\right). \quad (10)$$

This completes the proof.

3.5 Rényi entropy

The Rényi entropy is one of the most popular measures used to quantify the variability of random variable X. Entropy is a measure of system unpredictability that is frequently applied in areas including physics, cancer molecular imaging, and sparse kernel density estimation. In the case of an IPEP-distributed random variable, X, the Rényi entropy, for $R>0$ and $R\neq 1$ is given by:

$$I_R(x) = \frac{1}{1-R} log \left[\sum_{J=0}^{\alpha R-R} \frac{-(\alpha\beta\lambda)^R(-1)^j}{\lambda} \binom{\alpha R - R}{j} \left(1+x^{-\lambda}\right)^{-\beta j} B\left(\frac{R(\lambda+1)-1}{\lambda}, \frac{\lambda\beta(j+R)+\lambda R - R(\lambda+1)+1}{\lambda}\right) \right]$$

$$; R > 0 \text{ and } R \neq 1. \quad (11)$$

Proof

$$I_R(x) = \frac{1}{1-R} log \left[\int_0^\infty \left(f_{IPEP}(x;\alpha,\beta,\lambda)\right)^R dx \right],$$

then

$$I_R(x) = \frac{1}{1-R} log \left[\int_0^\infty \left(\alpha\beta\lambda x^{-\lambda-1}\left(1+x^{-\lambda}\right)^{-\beta-1}\left(1-\left(1+x^{-\lambda}\right)^{-\beta}\right)^{\alpha-1}\right)^R \right]$$

$$= \frac{1}{1-R} log \left[\int_0^\infty (\alpha\beta\lambda)^R x^{-R\lambda-R}\left(1+x^{-\lambda}\right)^{-R\beta-R}\left(1-\left(1+x^{-\lambda}\right)^{-\beta}\right)^{\alpha R-R} \right],$$

by using binomial expansion

$$\left(1-\left(1+x^{-\lambda}\right)^{-\beta}\right)^{\alpha R-R} = \sum_{J=0}^{\alpha R-R} (-1)^j \binom{\alpha R - R}{j} \left(1+x^{-\lambda}\right)^{-\beta j},$$

Then

$$I_R(x) = \frac{1}{1-R} log \left[\sum_{J=0}^{\alpha R-R} -(\alpha\beta\lambda)^R(-1)^j \binom{\alpha R - R}{j} \int_0^\infty x^{-R\lambda-R}\left(1+x^{-\lambda}\right)^{-\beta(j+R)-R} dx \right].$$

So.

$$I_R(x) = \sum_{J=0}^{\alpha R-R} (\alpha\beta\lambda)^R(-1)^j \binom{\alpha R - R}{j} \int_0^\infty x^{-R\lambda-R}\left(1+x^{-\lambda}\right)^{-\beta(j+R)-R} dx.$$

Thus, the computation of this integration

$$\int_0^\infty x^{-R\lambda-R}\left(1+x^{-\lambda}\right)^{-\beta(j+R)-R} dx = \frac{-1}{\lambda} B\left(\frac{R(\lambda+1)-1}{\lambda}, \frac{\lambda\beta(j+R)+\lambda R - R(\lambda+1)+1}{\lambda}\right).$$

This completes the proof.

3.6 Order Statistics

In the fields of reliability and life testing, order statistics have been extensively used. Order statistics play a significant role in many areas of statistical inference. Assume that X_1, X_2,...... , X_n is a random sample from the IPEP distribution. Let the corresponding order statistics be denoted by $X_{(1)}$, $X_{(2)}$,....... , $X_{(n)}$. The probability density function of the *rth* order statistics is provided by

$$f_{x(r)}(x) = \frac{1}{B(r,n-r+1)} \sum_{j=0}^{n-r} \sum_{i=0}^{j+r-1} \alpha\beta\lambda (-1)^{j+i} x^{-\lambda-1} \binom{n-r}{j} \binom{j+r-1}{i} \left(1+x^{-\lambda}\right)^{-\beta-1} \left(1-\left(1+x^{-\lambda}\right)^{-\beta}\right)^{\alpha(i+1)-1}. \quad (12)$$

Proof

$$f_{X(r)}(x) = \frac{f(x)}{B(r,n-r+1)} \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} [F(x)]^{j+r-1},$$

then

$$f_{x(r)}(X) = \frac{1}{B(r,n-r+1)}\sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j}\alpha\beta\lambda x^{-\lambda-1}\left(1+x^{-\lambda}\right)^{-\beta-1}\left(1-\left(1+x^{-\lambda}\right)^{-\beta}\right)^{\alpha-1}\left[1-\left(1-\left(1+x^{-\lambda}\right)^{-\beta}\right)^{\alpha}\right]^{j+r-1},$$

by using binomial expansion

$$\left[1-\left(1-\left(1+x^{-\lambda}\right)^{-\beta}\right)^{\alpha}\right]^{j+r-1} = \sum_{i=0}^{j+r-1}(-1)^i\binom{j+r-1}{i}\left(1-\left(1+x^{-\lambda}\right)^{-\beta}\right)^{\alpha i}.$$

This completes the proof.

3.7 Stress-strength parameter (SSP)

The measure of system performance (stress-strength reliability measure), defined by $R=P(Y<X)$, is given as the following; X represent the ability of the system under stress Y and assuming that X and Y

IPEP $(x;\alpha_1,\beta_1,\lambda)$,) and IPEP $(x;\alpha_2,\beta_2,\lambda)$, respectively.

$$R = 1 + \sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{\alpha_2}\alpha_1\beta_1\lambda(-1)^{i+j}\binom{\alpha_1-1}{i}\binom{\alpha_2}{j} B\left(1,\beta_1(i+1)-\beta_2 j\right). \quad (13)$$

Proof

$$R=\int_0^\infty f_{1IPEP}(x;\alpha_1,\beta_1,\lambda)F_{2IPEP}(x;\alpha_2,\beta_2,\lambda)dx,$$

then

$$R=\int_0^\infty \alpha_1\beta_1\lambda x^{-\lambda-1}\left(1+x^{-\lambda}\right)^{-\beta_1-1}\left(1-\left(1+x^{-\lambda}\right)^{-\beta_1}\right)^{\alpha_1-1}\left[1-\left(1-\left(1+x^{-\lambda}\right)^{-\beta_2}\right)^{\alpha_2}\right]dx,$$

$$=\int_0^\infty \alpha_1\beta_1\lambda x^{-\lambda-1}\left(1+x^{-\lambda}\right)^{-\beta_1-1}\left(1-\left(1+x^{-\lambda}\right)^{-\beta_1}\right)^{\alpha_1-1}dx - \int_0^\infty \alpha_1\beta_1\lambda x^{-\lambda-1}\left(1+x^{-\lambda}\right)^{-\beta_1-1}\left(1-\left(1+x^{-\lambda}\right)^{-\beta_1}\right)^{\alpha_1-1}\left(1-\left(1+x^{-\lambda}\right)^{-\beta_2}\right)^{\alpha_2}dx,$$

then

$$R = 1-\int_0^\infty \alpha_1\beta_1\lambda x^{-\lambda-1}\left(1+x^{-\lambda}\right)^{-\beta_1-1}\left(1-\left(1+x^{-\lambda}\right)^{-\beta_1}\right)^{\alpha_1-1}\left(1-\left(1+x^{-\lambda}\right)^{-\beta_2}\right)^{\alpha_2}dx.$$

By using binomial expansion

$$\left(1-\left(1+x^{-\lambda}\right)^{-\beta_1}\right)^{\alpha_1-1} = \sum_{i=0}^{\alpha_1-1}(-1)^i\binom{\alpha_1-1}{i}\left(1+x^{-\lambda}\right)^{-\beta_1 i}$$

and

$$\left(1-\left(1+x^{-\lambda}\right)^{-\beta_2}\right)^{\alpha_2} = \sum_{j=0}^{\alpha_2}(-1)^j\binom{\alpha_2}{j}\left(1+x^{-\lambda}\right)^{-\beta_2 j},$$

then

$$R = 1-\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{\alpha_2}\alpha_1\beta_1\lambda(-1)^{i+j}\binom{\alpha_1-1}{i}\binom{\alpha_2}{j}\int_0^\infty x^{-\lambda-1}\left(1+x^{-\lambda}\right)^{-\beta_1(i+1)-\beta_2 j}dx.$$

Thus, the computation of this integration

$$\int_0^\infty x^{-\lambda-1}\left(1+x^{-\lambda}\right)^{-\beta_1(i+1)-\beta_2 j}dx = \frac{-1}{\lambda}B(1,\beta_1(i+1)-\beta_2 j).$$

This completes the proof.

4. Method of Estimation

In this section, we will through how to estimate the IPEP distribution's parameters α, β and λ using the maximum likelihood method. We assume that $X=(x_1, x_2, \ldots, x_n)$ is a random sample of size n from the IPEP distribution with unknown values for α, β and λ.

4.1. Maximum Likelihood Estimation

The best known technique for parameter estimate is the method of maximum likelihood, see Casella and Berger [14]. Its effectiveness is due to a number of appealing characteristics, such as consistency, asymptotic performance, invariance, and intuitive appeal. Assuming that the observations from IPEP(α, β, λ) were used to generate a random sample of size n, the log likelihood function is provided by:

$$L(x,\alpha,\beta,\lambda)=\prod_{i=1}^{n}\alpha\beta\lambda x_i{}^{-\lambda-1}\left(1+x_i{}^{-\lambda}\right)^{-\beta-1}\left(1-\left(1+x_i{}^{-\lambda}\right)^{-\beta}\right)^{\alpha-1}$$

$$=\alpha^n\beta^n\lambda^n\prod_{i=1}^{n}x_i{}^{-\lambda-1}\left(1+x_i{}^{-\lambda}\right)^{-\beta-1}\left(1-\left(1+x_i{}^{-\lambda}\right)^{-\beta}\right)^{\alpha-1}.$$

(14)

By applying *ln* function on both sides,

$$\ell(x;\alpha,\beta,\lambda)=n\,ln\,\alpha+n\,ln\,\beta+n\,ln\,\lambda+-(\lambda+1)\sum_{i=0}^{n}ln(x_i)-(\beta+1)\sum_{i=0}^{n}ln\left(1+x_i{}^{-\lambda}\right)+(\alpha-1)\sum_{i=0}^{n}ln\left(1-\left(1+x_i{}^{-\lambda}\right)^{-\beta}\right)+(\alpha-1)\sum_{i=0}^{n}ln\left(1-\left(1+x_i{}^{-\lambda}\right)^{-\beta}\right).$$

We obtain the following normal equations by taking the first derivatives of ℓ(x; α, β, λ) with respect to the parameters α, β and λ parameters:

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=0}^{n}ln\left(1-\left(1+x_i{}^{-\lambda}\right)^{-\beta}\right),$$

(15)

$$\frac{\partial\ell}{\partial\beta}=\frac{n}{\beta}-\sum_{i=0}^{n}ln(1+x_i{}^{-\lambda})-\frac{(\alpha-1)\sum_{i=0}^{n}\left(1+x_i{}^{-\lambda}\right)^{-\beta}ln(1+x_i{}^{-\lambda})}{\sum_{i=0}^{n}(1-(1+x_i{}^{-\lambda})^{-\beta})}$$

(16)

and

$$\frac{\partial\ell}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=0}^{n}\ln(x_i)+\frac{(\beta+1)\sum_{i=0}^{n}x_i{}^{-\lambda}\ln(x_i)}{\sum_{i=0}^{n}\left(1+x_i{}^{-\lambda}\right)}-\frac{\beta(\alpha-1)\sum_{i=0}^{n}x_i{}^{-\lambda}ln(x_i)\left(1+x_i{}^{-\lambda}\right)^{-\beta}}{\sum_{i=0}^{n}\left(1+x_i{}^{-\lambda}\right)\left(1-\left(1+x_i{}^{-\lambda}\right)^{-\beta}\right)}.$$

(17)

By using the Newton-Raphson iteration approach for the solution of nonlinear equations after equating them to zero, it is possible to get the maximum likelihood estimators of the parameters α, β and λ.

4.2. Fisher's information matrix

The Fisher information is a measure use for mathematics statistics to assess the amount of knowledge that an observable random variable *X* knows about unidentified parameters of a distribution that models *X*. Formally, it is known as the score's variance or the expected value of the observable values. We utilized to make the information matrix in order to determine out an confidence intervals for the parameters. I_n (α, β, λ) is the corresponding 3×3 observed information matrix.

$$I_n=\begin{pmatrix}\frac{\partial^2\ell}{\partial\alpha^2} & \frac{\partial^2\ell}{\partial\alpha\partial\beta} & \frac{\partial^2\ell}{\partial\alpha\partial\lambda}\\ \frac{\partial^2\ell}{\partial\beta\partial\alpha} & \frac{\partial^2\ell}{\partial\beta^2} & \frac{\partial^2\ell}{\partial\beta\partial\lambda}\\ \frac{\partial^2\ell}{\partial\lambda\partial\alpha} & \frac{\partial^2\ell}{\partial\lambda\partial\beta} & \frac{\partial^2\ell}{\partial\lambda^2}\end{pmatrix}.$$

The elements of I_n are given by

$$I_{1,1}=\frac{\partial^2\ell}{\partial\alpha^2}=\frac{-n}{\alpha^2},$$

(23)

$$I_{1,2}=\frac{\partial^2\ell}{\partial\alpha\partial\beta}=\frac{-\sum_{i=0}^{n}\left(1+x_i{}^{-\lambda}\right)^{-\beta}ln\left(1+x_i{}^{-\lambda}\right)}{\sum_{i=0}^{n}(1-(1+x_i{}^{-\lambda})^{-\beta})},$$

(24)

$$I_{1,3}=\frac{\partial^2\ell}{\partial\alpha\partial\lambda}=\frac{\beta\sum_{i=0}^{n}x_i^{-\lambda}ln(x_i)\left(1+x_i^{-\lambda}\right)^{-\beta-1}}{\sum_{i=0}^{n}(1-(1+x_i^{-\lambda})^{-\beta})},$$

(25)

$$I_{2,2}=\frac{\partial^2\ell}{\partial\beta^2}=\frac{-n}{\beta^2}-\frac{(\alpha-1)\sum_{i=0}^{n}\left(1+x_i^{-\lambda}\right)^{-\beta}ln\left(1+x_i^{-\lambda}\right)^2}{\sum_{i=0}^{n}\left(1-\left(1+x_i^{-\lambda}\right)^{-\beta}\right)}.$$

*(26)*and

$$I_{3,3}=\frac{\partial^2\ell}{\partial\lambda^2}=\frac{-n}{\lambda^2}-\frac{(\beta+1)\sum_{i=0}^{n}x_i^{-\lambda}ln(x_i^2)}{\sum_{i=0}^{n}\left(1+x_i^{-\lambda}\right)^2}+\frac{\beta(\alpha-1)\sum_{i=0}^{n}x_i^{-\lambda}ln(x_i^2)\left(1+x_i^{-\lambda}\right)^{-\beta}-\beta\sum_{i=0}^{n}\left(1+x_i^{-\lambda}\right)-\sum_{i=0}^{n}x_i}{\sum_{i=0}^{n}\left(1-\left(1+x_i^{-\lambda}\right)^{-\beta}\right)} \ldots (27)$$

4.3 Approximate Confidence Intervals

It is difficult to determine the exact confidence intervals, since the MLEs can't be determined in closed forms. In order to derive the asymptotic confidence intervals for the model parameters, we can use of the MLEs asymptotic behavior. The asymptotic distribution of the observed information matrix's inverse approximated using a large sample. The approximated variance covariance matrix of the parameters presented as follows:

$$\Delta^{-1}=\begin{pmatrix}\frac{\partial^2\ell}{\partial\alpha^2} & \frac{\partial^2\ell}{\partial\alpha\partial\beta} & \frac{\partial^2\ell}{\partial\alpha\partial\lambda}\\ \frac{\partial^2\ell}{\partial\beta\partial\alpha} & \frac{\partial^2\ell}{\partial\beta^2} & \frac{\partial^2\ell}{\partial\beta\partial\lambda}\\ \frac{\partial^2\ell}{\partial\lambda\partial\alpha} & \frac{\partial^2\ell}{\partial\lambda\partial\beta} & \frac{\partial^2\ell}{\partial\lambda^2}\end{pmatrix}^{-1}$$

$$=\begin{pmatrix}var(\widehat{\alpha}) & var(\hat{\alpha},\hat{\beta}) & var(\hat{\alpha},\hat{\lambda})\\ var(\hat{\beta},\hat{\alpha}) & var(\widehat{\beta}) & var(\hat{\beta},\hat{\lambda})\\ var(\hat{\lambda},\hat{\alpha}) & var(\hat{\lambda},\hat{\beta}) & var(\widehat{\lambda})\end{pmatrix}.$$

Hence, the asymptotic (1-γ) 100% confidence intervals for α, β and λ are given by

$$\left\{\hat{\alpha}_L=\hat{\alpha}-z_{\frac{\gamma}{2}}\sqrt{var(\widehat{\alpha})},\hat{\alpha}_U=\hat{\alpha}+z_{\frac{\gamma}{2}}\sqrt{var(\widehat{\alpha})}\right\},$$

$$\left\{\hat{\beta}_L=\hat{\beta}-z_{\frac{\gamma}{2}}\sqrt{var(\widehat{\beta})},\hat{\beta}_U=\hat{\beta}+z_{\frac{\gamma}{2}}\sqrt{var(\widehat{\beta})}\right\}$$

and

$$\left\{\hat{\lambda}_L=\hat{\lambda}-z_{\frac{\gamma}{2}}\sqrt{var(\widehat{\lambda})},\hat{\lambda}_U=\hat{\lambda}+z_{\frac{\gamma}{2}}\sqrt{var(\widehat{\lambda})}\right\}.$$

Where, $z_{\frac{\gamma}{2}}$ is the upper (1-γ) 100% percentile of the standard normal distribution.

5. Applications

In this section, we investigate the capability of the Inverse Power Exponentiated Pareto distribution (IPEPD) by fitting distributions, namely, the Inverse Power Lindely (IPL), Exponentiated Inverse Power Lindely (ExpIPL), Exponential (Exp), the Gompertz (Gz), Lindely (Lin), weighted Gompertz (W-Gz) and weighted Lindley (W-L), to two data sets. The Inverse Power Exponentiated Pareto distribution explain much flexibility than the corresponding distributions. By making use of real data set, we illustrate the applicability of the IPEP distribution among a set of classical and recent distributions, based on a set of goodness-of-fit statistics. We use the maximum likelihood method to estimate the model parameters. We compared the models' goodness-of-fit with the Akaike Information Criterion (AIC), consistent Akaike information criterion (CAIC), the corrected Akaike information criterion (AICC), Bayesian Information Criterion (BIC) and Hannan-Quinn Information Criterion (HQIC) goodness -of-fit statistics. Further, we get the Kolmogorov-Smirnov (K-S) statistic with its corresponding P-value. In general, the model has the smaller values of these statistics and the largest value of the P-value is the best model to fit the data.

5.1. Growth hormone data

This dataset makes up 35 observations of the Growth hormone data set. Growth hormone deficit was observed in children participating in the program hormonal (de Crescimento da Secretaria da Sa'ude de Minas Gerais). The data includes an estimate of how long it took the children to grow to their desired height after starting growth hormonal treatment. The data set have been analyzed by Lemos de Morais [15].

Table 1: Growth hormone data.

2.15	2.20	2.55	2.56	2.63
2.74	2.81	2.90	3.05	3.41
3.43	3.43	3.84	4.36	4.42
4.51	4.60	4.61	4.75	5.03
5.10	5.44	5.90	5.96	6.77
7.82	8.00	8.16	8.21	8.72
10.40	13.20	13.70		

Table 2: Estimates of models for growth hormone data set

	Estimates			Statistics						
Model	α	β	λ	AIC	BIC	AICC	CAIC	HQIC	K-S	P-value
IPEP (α, β, λ)	1.34	21.6	2.13	161.43	166.095	162.20	162.20	163.04	0.084	0.95
IPLin (α, λ)	2.62		0.12	161.45	166.11	162.23	162.23	163.06	0.097	0.86
Exp(λ)			0.19	188.82	190.37	188.94	188.94	189.35	0.33	0.0008
Gz (β, λ)		0.50	0.18	178.20	181.30	178.57	178.57	179.27	0.21	0.10
Lin(λ)			0.33	176.95	178.50	177.07	177.07	177.48	0.25	0.03
W-Gz (α, β, λ)	5.57	13.31	0.03	164.69	169.35	165.46	162.70	163.40	0.10	0.89
W-Lin (α, λ)	4.18		0.56	162.33	165.44	162.70	162.70	163.40	0.10	0.88

From Table 2, the smallest values of the K-S, AIC, BIC and HQIC and the greatest value of the p-value are observed for the IPEP distribution. Therefore, we reached the conclusion that the IPEP distribution offers the best match when compared to the other distributions. The estimated densities function for the comparable distributions of the data set are presented based on the density function of each distribution in Fig. 6, which supports this result.

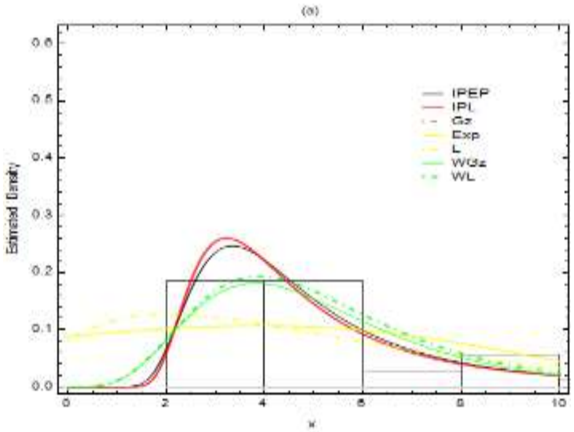

Fig. 5: Estimated probability density function for the considered distributions for the growth hormone data.

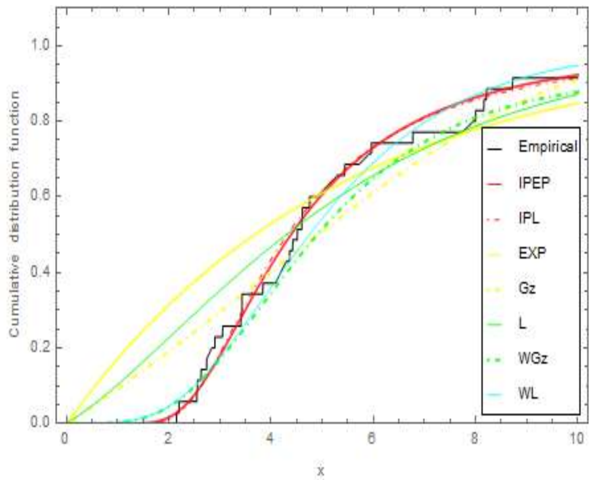

Fig. 6: Estimated Cumulative distribution functions for the considered distributions for the growth hormone data

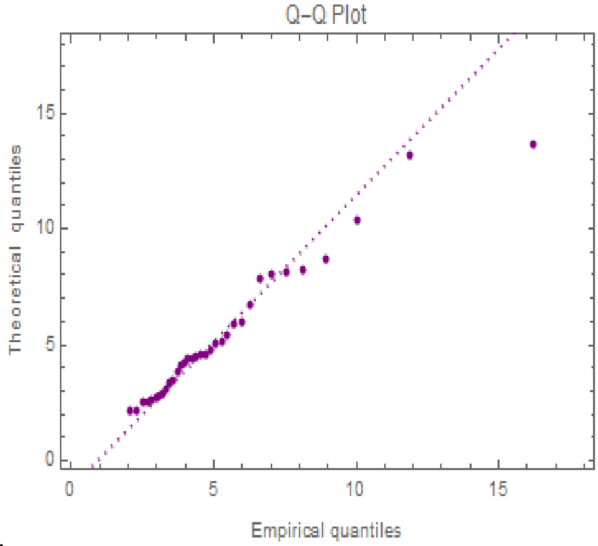

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Fig. 7: represents the empirical quantile function of IPEP distribution
for the growth hormone data.

5.2. Flood levels data

The next dataset, which represents the maximum flood levels (in million cubic feet/s) of the Susquehanna River at Harrisburg, Pennsylvania from 1890 to 1969, was given by Dumonceaux and Antle [16]. Maswadah [17] has examined into these data, and the results are as follows:

Table 3: Flood levels data.

0.645	0.613	0.315	0.449	0.297
0.402	0.379	0.423	0.379	0.324
0.269	0.740	0.218	0.412	0.494
0.416	0.338	0.392	0.484	0.265

Table 4: Estimates of models for Flood levels data set.										
	Estimates			Statistics						
Model	α	β	λ	AIC	BIC	AICC	CAIC	HQIC	K-S	P-value
IPEP (α, β, λ)	118.95	3.64	1.26	-20.12	-17.13	-18.62	-18.6	-19.54	0.15	0.698
IPLin (α, λ)	2.62		0.12	-23.23	-21.24	-22.52	-22.8	-22.5	0.15	0.6696
ExpIPLin (α, β, λ)	3.017	0.28	0.19	-21.42	- 18.44	-19.92	-20.8	-19.9	0.15	0.669
(Exp(λ			2.36	7.60	8.59	7.82	7.82	7.79	0.47	0.0003
Gz (β, λ)	0.05	0.05	6.38	-16.66	-14.67	-15.96	-15.95	16.27	0.21	0.32
Lin(λ)			2.96	6.36	7.36	¢.58	6.58	6.55	0.45	0.0005

From Table 4, the greatest value of the p-value and the smallest values of the K-S, AIC, BIC and HQIC are obtained for the IPEP distribution. Therefore, we reach the conclusion that the IPEP distribution offers the best match when compared to the other distributions. The estimated densities function for the comparable distributions of the data set are presented based on the density function of each distribution in Fig. 8, which supports this conclusion.

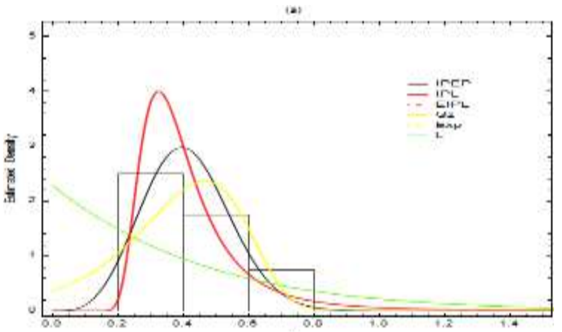

Fig. 8: Estimated densities functions for the considered distributions for the Flood levels data.

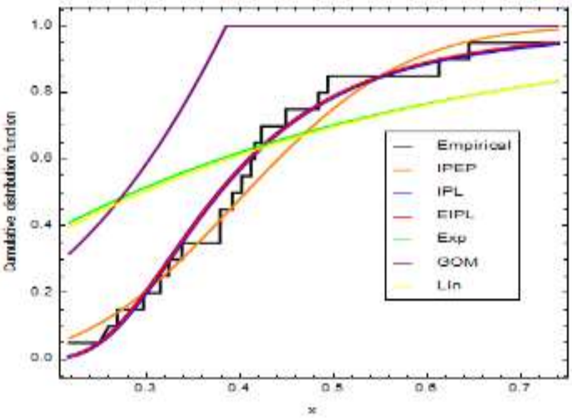

Fig. 9: Estimated Cumulative distribution functions for the considered distributions for the Flood levels data.

6. Simulation Study

This section explores the performance and behavior of several estimation techniques used to estimate the IPEP parameters applying extensive simulation data. In order to, multiple sample sizes

n= {20,40,50,100,150,200} and several values of the parameters α, β and λ, α=10, β=1.8 and λ=0.5, α=5, β=1.5 and λ=0.5 are considered. The MSE can be determined by the following equation:

$$MSE = \frac{1}{N}\sum_{i=0}^{N}\left(\theta - \hat{\theta}\right)^2,$$

where θ= (α, β, λ).

Table. 5 MSE of the parameters of the IPEP along with Average BIAS, Average MSE and Average interval length of														
β	λ	n	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\lambda}$			
			MLE	BIAS	MSE	AICS	MLE	BIAS	MSE	AICS	MLE	BIAS	MSE	AICS
1.8	0.5	20	0.898	-9.102	82.98	54.69	1.667	-0.133	0.101	20.187	1.033	0.533	0.35	7722.51
		30	20.82	-9.178	84.35	9.222	1.613	-0.187	0.1007	2.186	1.072	0.571	0.379	4.6867
		50	0.900	-9.099	82.96	77.37	1.594	-0.205	0.0826	15.712	1.107	0.607	0.414	9126.63
		100	0.854	-9.146	83.79	23.98	1.537	-0.263	0.0963	4.2198	1.122	0.622	0.421	9126.63
		150	0.887	-9.112	83.18	10.35	1.537	-0.263	0.0854	2.199	1.125	0.625	0.422	13.4009
		200	0.913	-9.088	82.69	7.538	1.541	-0.259	0.0814	1.6807	1.106	0.605	0.389	4.2661
1.5	0.5	20	0.712	-4.288	18.53	10.16	1.364	-0.136	0.0827	2.4309	1.017	0.517	0.342	55.1627
		30	0.783	-4.217	17.94	114.9	1.311	-0.189	17.936	114.97	0.974	0.475	0.281	4257.57
		50	0.776	-4.224	17.99	104.0	1.279	-0.221	0.0728	14.003	0.967	0.467	0.264	9698.3
		100	0.831	-4.169	17.55	14.38	1.285	-0.215	0.0637	2.6623	0.962	0.462	0.256	58.7897
		150	0.856	-4.143	17.34	12.24	1.289	-0.211	0.0618	2.1775	0.939	0.439	0.241	27.4508
		200	0.844	-4.156	17.41	5.740	1.302	-0.197	0.0496	1.1566	0.941	0.441	0.234	3.0504

Table. 6 MSE of S(t) and h(t) of the IPEP along with Average BIAS, Average MSE and Average interval length of AICS.											
α	β	λ	n	$\widehat{S(t)}$				$\widehat{h(t)}$			
				MLE	BIAS	MSE	AICS	MLE	BIAS	MSE	AICS
10	1.8	0.5	20	0.9817	0.0024	0.0003	0.0957	0.2631	-0.0059	0.0361	0.9695
			30	0.9824	0.0031	0.0003	0.0603	0.259	-0.01	0.0297	0.6588
			50	0.9856	0.0063	0.0002	0.0764	0.2236	-0.0454	0.0211	1.3184
			100	0.9852	0.0059	0.0001	0.0463	0.2332	-0.0358	0.0134	0.5874
			150	0.9866	0.0073	0.0001	0.0309	0.2178	-0.0512	0.0114	0.3704
			200	0.9868	0.0075	0.0001	0.0238	0.0238	-0.0511	0.0078	0.3012
5	1.5	0.5	20	0.9458	0.0082	0.002	0.1469	0.5329	-0.018	0.1067	1.0776
			30	0.9419	0|0044	0.0011	0.3736	0.5594	0.0085	0.0516	1.6882
			50	0.9386	0.0011	0.0008	0.3232	0.5883	0.0374	0.0403	1.7664
			100	0.9436	0.006	0.0004	0.099	0.5617	0.0108	0.0166	0.5763
			150	0.941	0.0035	0.0003	0.0835	0.5837	0.0329	0.0137	0.4485
			200	0.9447	0.0072	0.0003	0.0543	0.5619	0.0111	0.011	0.3702

6. Concluding remarks

In this paper, we propose the inverse-power Exponentiated Pareto (IPEP) model, a more accurate and flexible extension of the Exponentiated Pareto distribution for fitting engineering and medical data. Based on the inverse-power transformation method, the new model was constructed. Based on the parameters of its shape, the hazard rate function of the IPEP distribution can take on the following shapes: bathtub-shaped, monotonously ascending, declining, and upside down. As a result, it can be effectively expanded to lifetime data analysis. Maximum likelihood estimation is used to estimate the three parameters of the IPEP distribution, and some of its mathematical characteristics are obtained. The results of the simulation are used to investigate the behavior and performance of various estimators and

We are going to find estimation of the parameters of the inverse-power Exponentiated Pareto (IPEP) model under progressive type II censored data, also we will make multicomponent stress-strength under progressive type II censored data and the performance index for the inverse-power Exponentiated Pareto distribution.

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